Dynamical Systems

Exam 19 January 2026

university of groningen

The exam consists of 4 questions. You have 120 minutes to do the exam. You can achieve 50 points in total which includes a bonus of 5 points.

1. [**6+3=9 Points**]

   (a) For each of the following bifurcations, give an example of a one-parameter family of one-dimensional time-continuous systems showing the respective bifurcation, plot the bifurcation diagram and describe in words the bifurcation scenario.

      i. Transcritical bifurcation

      ii. Pitchfork bifurcation

   (b) For the family of planar system

   $$r' = r^3 - r,$$
   $$\theta' = \cos\theta - a,$$

   where r and θ are polar coordinates and $a \in \mathbb{R}$ is a parameter, show the bifurcation diagram in the form θ versus a and plot representative phase portraits in Cartesian coordinates.

2. [**7 Points**]

   For a C^∞ map $f : \mathbb{R}^2 \to \mathbb{R}$, consider the family of discrete-time systems $x_{n+1} = f(x_n, a)$, $n \in \mathbb{Z}_{\geq 0}$, with parameter $a \in \mathbb{R}$. Suppose that for fixed $x_0, a_0 \in \mathbb{R}$,

   (i) $f(x_0, a_0) = x_0$,

   (ii) $\frac{\partial f}{\partial x}(x_0, a_0) = 1$,

   (iii) $\frac{\partial^2 f}{\partial x^2}(x_0, a_0) \neq 0$, and

   (iv) $\frac{\partial f}{\partial a}(x_0, a_0) \neq 0$.

   Show that the discrete-time system has a saddle-node bifurcation at (x_0, a_0). (Hint: make use of the Implicit Function Theorem)

– please turn over –

3. **[4+3+4+3=14 Points]**

Consider the planar system

$$x' = y,$$
$$y' = -\nu y + x^2 - 1,$$

where $\nu \geq 0$ is a parameter.

(a) Show that the system has the two equilibrium points $(x_-, y_-) = (-1, 0)$ and $(x_+, y_+) = (1, 0)$, and determine their stability from the linearization.

(b) Show that for $\nu = 0$, the system is Hamiltonian with Hamilton function

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + x + \frac{2}{3}$$

and sketch the phase portrait in the (x, y) plane.

(c) Show that for $\nu \geq 0$ and each $0 < h < 4/3$, H is a Lyapunov function in the region $D_h = \{(x, y) \in \mathbb{R}^2 \mid H(x, y) \leq h,\ x < 1\}$ and use the Lasalle Invariance Principle to show that for $\nu > 0$, the equilibrium at $(x_-, y_-) = (-1, 0)$ is asymptotically stable with D_h belonging to the basin of attraction.

(d) Sketch the phase portrait for $\nu > 0$ by paying attention to the stable and unstable curves of the saddle at $(x_+, y_+) = (1, 0)$. What can you say about the basin of attraction of $(x_-, y_-) = (-1, 0)$.

4. **[9+6=15 Points]**

(a) Consider the map

$$t : [-1, 1] \to [-1, 1], \quad x \mapsto 1 - 2|x|.$$

Plot the graph of t, its second iterate t^2 and its third iterate t^3 and show by direct proof (i.e. without using a conjugacy) that the discrete-time system $x_{n+1} = t(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, satisfies all three conditions of Devaney's definition of chaos.

(b) Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be compact intervals and suppose the two discrete-time systems $x_{n+1} = f(x_n)$ and $y_{n+1} = g(y_n)$ defined by maps $f : I \to I$ and $g : J \to J$ are topologically conjugate. Show that if the discrete-time system $x_{n+1} = f(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, is topologically transitive, then the discrete-time system $y_{n+1} = g(y_n)$, $n \in \mathbb{Z}_{\geq 0}$, is also topologically transitive.